Preprint TK2004b

VORTICITY IN FLOW FIELDS

in relation to Prandtl's work and subsequent developments

Tsutomu Kambe
IDS, Higashi-yama 2-11-3, Meguro-ku, Tokyo 153-0043, Japan
e-mail:*kambe@gate01.com*

Abstract: One of the important properties of boundary layer is, from the point of view of *Strömungslehre*, that it pumps vorticity into the flow field. Vorticity in fluid flows plays diverse roles and produces tremendous variety of flow phenomena. We consider four aspects of vorticity in fluid flows: (A) Kinematical aspect of a vortex sheet, (B) mechanical aspect of hydrodynamic impulse of a vortex system, (C) dynamical aspect: vorticity dynamics, excitation of acoustic waves and formation of dissipative structure in turbulence, and (D) gauge field associated with local rotational symmetry.

Key words: vorticity, impulse, acoustic source, dissipative structure, gauge field

1. INTRODUCTION

Boundary layer is a transition layer of velocity adjacent to a solid surface, and it is the place where vorticity is created. Boundary layer can be separated from the wall. Separated layer with vorticity is a common source of the vorticity in flow fields. All these essential properties of the boundary layer were established in the celebrated paper [**1**] of Prandtl in 1904.

Given a velocity field $\boldsymbol{v}(\boldsymbol{x})$, the vorticity $\boldsymbol{\omega}$ is defined by $\boldsymbol{\omega} = \mathrm{rot}\, \boldsymbol{v}$, which is a measure of local rotation of a fluid element. From the point of view of *Strömungslehre*, one of the important characteristics of the boundary layer is that it pumps vorticity into the flow field. Prandtl explained in the same paper [**1**] that the boundary layer is prone to be separated from the solid wall when pressure increases along the direction of flow, and showed some experimental evidence in the case of flows around a circular cylinder. Along a cylinder in a free stream, the pressure increases over the rear part of the cylinder surface. The fluid decelerated by the viscosity in the boundary layer is forced to separate from the wall by the adverse pressure gradient under combined action of the forward free stream.

By the separation, the vorticity $\boldsymbol{\omega}$ in the boundary layer is transported to the interior field ([**2**]: II §6 and III §6). Separation of boundary layer is most common source of vortices in fluid flows of uniform density at a high Reynolds number (under a conseravtive force), where there is no interior mechanism of creation of vortices. In §4.1, we will consider some mechanisms of creation of vorticity. The separation occurs most effectively at a sharp edge of a body.

Fluid flows with non-vanishing vorticity are called *rotational* flows. There are an infinite variety of rotational flows. In fact, given a simply connected bounded domain of flows together with boundary conditions for the velocity, an *irrotational* flow is determined uniquely. However, rotational flow is not unique, and one can conceive all sorts of complex flows, depending on the vorticity distribution under the same boundary conditions.

In this review article, we are going to consider various roles of vorticity in fluid flows. (A) Kinematical aspect: a vortex sheet, *i.e.* a thin vorticity layer sandwiched between two irrotational flows, is a transition layer of discontinuity in velocity. This is described as a review of Prandtl's work in §2. (B) Mechanical aspect: A system of vorticity (i.e. a certain distribution of vorticity) possesses a certain amount of momentum (and also some energy). Therefore, a body shedding a set of vortices from its surface is subject to a reaction force, which is described in §3. (C) Dynamical aspect: (i) Creation of vorticity is considered on the basis of the evolution equation of vorticity, (ii) nonlinear interaction between $\boldsymbol{\omega}$ and $\boldsymbol{v}$ excites density waves, resulting in generation of acoustic waves, and (iii) the vorticity is an agent forming dissipative structures in turbulence which are visualized as fine-scale slender objects characterized with high level of vorticity magnitude. (D) Gauge field: Vorticity is regarded as a gauge field, which is defined in the variational formulation by requiring that the equation of motion should be invariant with respect to local rotational gauge transformation.

2. KINEMATICAL ASPECT (A review of Prandtl's work)

There are two different ways nominally, in which vortices are brought to interior of flows of a fluid of small viscosity. First, when a fluid flows round a bluff body such as a circular cylinder, the boundary layer separates at a certain point on the body surface and penetrates into the interior of fluid. Secondly, when the fluid flows around a sharp edge, the streams along the two sidewalls of the edge separate at the edge. This forms a surface of separation, which is a discontinuity surface between two velocities v_1 and v_2 on the two sides. On account of viscosity, this separation is regarded as merging of two boundary layers on both sides of the edge, and the velocity profile within the layer takes a form of a function *tanh* expressing transition from one velocity v_1 to another v_2. This layer is obviously rotational, i.e. it has nonzero vorticity. In the limit of vanishing viscosity, this layer tends to a surface, called a *vortex sheet*. Thus, the thin vorticity layer (sandwiched between two irrotational flows) represents a transition layer of *velocity discontinuity*.

Two conditions must always be satisfied on the surface of discontinuity. First, this surface must be a material surface consisting of the same fluid particle. Secondly, the pressure must be continuous across the vortex sheet. Separation at a sharp edge can be interpreted as the same mechanism that is working over the surface of a circular cylinder. For, a sharp edge can be regarded as a limiting surface when one of the radii of curvature of the wall surface becomes very small ([**3**]: §4.8). The flow along the edge will encounter a sudden increase of pressure after passing by the edge.

The vortex sheet (the pressure being continuous across it) is unstable and tends to rollup into a sequence of eddies, called the Kelvin-Helmholtz instability [**6**]. This occurs as a result of imbalance of pressure across the sheet when small wavy perturbations are imposed ([**2**], II § 6; [**4**], §94).

Starting vortex is another well-known example, which is shed from a sharp trailing edge of an aerofoil driven impulsively ([**4**], §93). After a short time, a vortex is left behind, and the aerofoil aquires a circulation round itself which is equal and opposite to that of the departed vortex.

3. MECHANICAL ASPECT

Suppose that a vorticity field is given by $\boldsymbol{\omega}(\boldsymbol{x}, t)$ at a spatial point $\boldsymbol{x}$ at a time t. The vorticity field is characterized by a hydrodynamic impulse when the fluid density ρ is assumed constant. The impulse $\boldsymbol{P}$ is defined by

$$\boldsymbol{P} = A_n \, \rho \int \boldsymbol{x} \times \boldsymbol{\omega}(\boldsymbol{x}, t) \; \mathrm{d}^n \boldsymbol{x}, \tag{1}$$

where $A_n = 1$ or $1/2$ according as the $\boldsymbol{\omega}$ distribution is 2D ($n = 2$) or 3D ($n = 3$), respectively. The impulse $\boldsymbol{P}$ is interpreted by the impulsive force $\boldsymbol{f}(\boldsymbol{x}) \, \delta(t - t_0)$ necessary to generate instantaneously the motion from rest at $t = t_0$. It can be shown that $\boldsymbol{P} = \int \boldsymbol{f}(\boldsymbol{x}) \, \mathrm{d}^n \boldsymbol{x}$ [**6**]. The impulse represents an effective total momentum, because $\boldsymbol{P}$ satisfies the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{P} = \int \boldsymbol{F}_r \; \mathrm{d}^n \boldsymbol{x},$$

where $\boldsymbol{F}_r$ denotes the non-conservative body force, i.e. $\mathrm{rot} \boldsymbol{F}_r \neq 0$ (viscoisty has no effect on the rate of change of $\boldsymbol{P}$ in unbounded fluid flows [**6**]). Hence, if the applied force has a potential (then $\boldsymbol{F}_r = 0$), the $\boldsymbol{P}$ is invariant.

Figure 1 is taken from the book [**5**] (Plate 22, Fig.55). This is a most impressive photograph, illustrating the impulse of a vortex system. The vortices are shed by an aerofoil that has been started impulsively from rest to a steady motion, and stopped suddenly shortly after. During the motion, the aerofoil gained a lift, *i.e.* an upward momentum. As a reaction, the fluid acquired a downward momentum, which is represented by the vortex pattern called a vortex pair, regarded as an object carrying some amount of downward momentum. Assuming that the flow field of Fig.1 is composed of two 2D vortices, and denoting the strength of the right vortex with $+\Gamma$ and the left by $-\Gamma$ and their separation distance with d, the magnitude of impulse of the two-vortex system is given by $P = \rho\Gamma d$ from Eq.(1), which is directed downward.

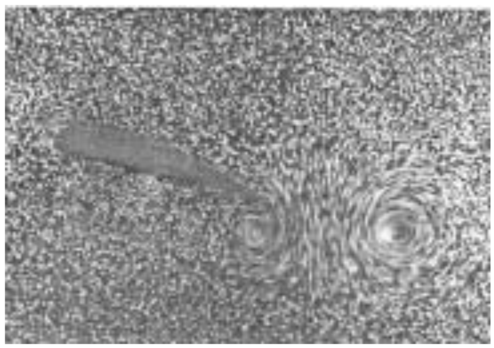


Figure 1. A vortex pair shed by an aerofoil [**5**].

Figure 2 shows a vortex ring generated by a shock wave (a circular arc on the right) emerging from a nozzle (a vertical dark shade on the left end) of circular cross-section. The generation mechanism is attributed to the impulse of the shock wave coming out impulsively to the open space. Denoting the vortex strength with Γ and its ring radius with r, and assuming axisymmetry, magnitude of the impulse of a circular vortex ring defined with Eq.(1) is given by $P = \pi\rho\Gamma r^2$, which is directed toward right.

A solid body shedding a set of vorticities from its surface is subject to a reaction force. If the reaction force is perpendicular to the direction of motion of the body, it is felt as a lift (or lateral) force. If the reaction is in the opposing direction of body's motion, then it is a drag. Drag of a bluff body can be interpreted partly by this sort of vortex drag [**8**]. On the other hand, if the reaction is in the same direction as the body's motion, then it is felt as a thrust. Animals are wise enoug to take advantage of the reaction forces variously.[1] A pair of tip vortices leaving a finite wing in steady

[1] *e.g.* Taylor GK, Nudds RL & Thomas ALR. "Flying and swimming animals cruise at

rectilinear motion can be interpreted in this context ([**5**]: Chap.VI, C).

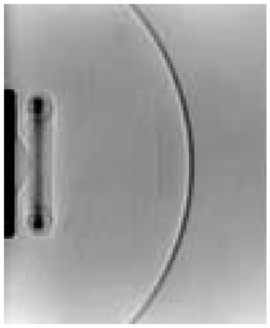


Figure 2. A vortex ring generated by a shock impulse [**7**].

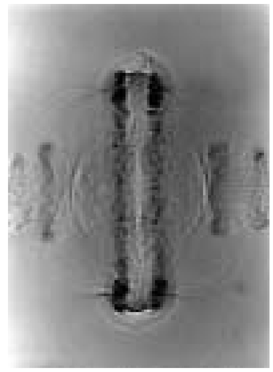


Figure 3. Shadowgraph of head-on collision of two vortex rings [**15**].

4. DYNAMICAL ASPECT

4.1 Creation of vorticity

Euler's equation of motion for a compressible ideal fluid moving under an external force $\boldsymbol{F}$ per unit mass is

$$\partial_t \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{v} + \nabla(\tfrac{1}{2}\, v^2) = -\rho^{-1}\, \nabla p + \boldsymbol{F}, \tag{2}$$

where p is the pressure (the density ρ is not necessarily constant). Taking rot of (2), we obtain an evolution equation for $\boldsymbol{\omega} = \mathrm{rot}\, \boldsymbol{v}$ (see *e.g.* [**6**]):

$$\partial_t \boldsymbol{\omega} + \mathrm{rot}(\boldsymbol{\omega} \times \boldsymbol{v}) = \rho^{-2} \nabla \rho \times \nabla p + \mathrm{rot}\, \boldsymbol{F}_r, \tag{3}$$

where $\boldsymbol{F}_r$ is the non-conservative part of $\boldsymbol{F}$. If the fluid is *barotropic*, *i.e.* $p = p(\rho)$, then $\nabla \rho \times \nabla p = p'(\rho)\, \nabla \rho \times \nabla \rho = 0$. In addition if the force is conservative, $\boldsymbol{F}_r = 0$. Then, the right hand side vanishes and we obtain

$$\partial_t \boldsymbol{\omega} + \mathrm{rot}(\boldsymbol{\omega} \times \boldsymbol{v}) = 0. \tag{4}$$

Based on this equation, Helmholtz's three laws of vortex motion are derived [**4, 6**]: (i) Persistence of irrotationality, (ii) material line remains a vortex line, and (iii) strength of an infinitely thin vortex tube is invariant when the vortex moves.[2] Furthermore, the Kelvin's circulation theorem is also derived

a Strouhal number tuned for high power efficiency", *Nature* **425** (16 Oct 2003), 707-711.

[2]Helmholtz originally assumed $\mathrm{div}\, \boldsymbol{v} = 0$.

by using (2) with $p = p(\rho)$ and $\boldsymbol{F} = \mathrm{grad}\,\Psi$ (conservative body force with a potential Ψ). The equation (4) describes essentially that the vorticity $\boldsymbol{\omega}$ is *frozen* to the fluid flow of velocity $\boldsymbol{v}$, and that no vorticity is created in the evolution governed by (4).

Putting it the other way, the right hand side of Eq.(3) states that the vorticity is created in two ways. The pressure p in general depends on two thermodynamic variables $p = p(\rho, s)$ (say), where s is the entropy per unit mass, and $\nabla\rho \times \nabla p \neq 0$. Then, the vorticity is created by the first term. This is called the *baroclinic* effect. Bjerkness [**9**] gave a geometrical interpretation of the creation of circulation by this term [**4**, §85]. In addition, if the force is non-conservative (then $\boldsymbol{F}_r \neq 0$), the second term also can generate vorticity.

A body moving relative to fluid can be replaced kinematically by a distribution of image vorticity within the body. In steady motion, the distribution of vorticity is fixed relative to the body and is referred to as *bound* vorticity. It does not in general satisfy the Helmholtz laws. Vorticity satisfying the Helmholtz laws is referred to as *free* vorticity. A typical example of the bound vortex is the Prandtl's lifting vortex of strength Γ in a stream of velocity U and density ρ. The lift L acting on the bound vortex per unit length is given by $L = \rho U \Gamma$ (Kutta (1902), Joukowsky (1906)) [**5**, §98; **6** §3.1]. Given the circulation Γ, the lift is independent of the shape of the body. A lifting vortex is not a physical reality, but a very useful concept for the theory of lift.

4.2 Exciting acoustic waves

An acoustic wave is generated by a localized rotational flow $\boldsymbol{v}$ (with a localized vorticity $\boldsymbol{\omega}$). At a low Mach number, the sound source is identified with a term of the form $\rho_0\,\mathrm{div}(\boldsymbol{\omega} \times \boldsymbol{v})$, by Powell [**10**] and Howe [**11**], where ρ_0 is the undisturbed fluid density. Thus, the wave equation for the acoustic pressure p' is written approximately as

$$c^{-2}\,\partial_t^2 p' - \nabla^2 p' = \rho_0\,\mathrm{div}(\boldsymbol{\omega} \times \boldsymbol{v}), \tag{5}$$

in the limit of $M (= U/c) \to 0$, where c is the sound speed and U a representative flow velocity. This is obtained in the following way. From the fundamental conservation equations of mass, momentum and energy for flow of an ideal fluid of uniform entropy, one can derive [**14**, Appendix A]

$$\left(c^{-2}\,\partial_t^2 - \nabla^2\right)\left(h + \tfrac{1}{2}\,\boldsymbol{v}^2\right) = \mathrm{div}(\boldsymbol{\omega} \times \boldsymbol{v}), \qquad = S(\boldsymbol{x}, t), \tag{6}$$

where h is the entalpy per unit mass. It is assumed that the source flow $\boldsymbol{v}(\boldsymbol{x}, t)$ is locallized in space and its representative Mach number M is sufficiently low. Then, the wave equation (6) can be transformed to an integral form,

$$h(\boldsymbol{x}, t) + \tfrac{1}{2}\,\boldsymbol{v}^2(\boldsymbol{x}, t) = (1/4\pi) \int S(\boldsymbol{y}, t_{\mathrm{r}})\,\mathrm{d}^3\boldsymbol{y}.$$

where the wave is observed at $\boldsymbol{x}$ and the source is located at $\boldsymbol{y}$, and $t_{\mathrm{r}} = t - |\boldsymbol{x} - \boldsymbol{y}|/c$ is the retarded time. Suppose that the observation is made in such a far field as $|\boldsymbol{x}| \to \infty$ and $h + \frac{1}{2}\boldsymbol{v}^2 \to p'/\rho_0$ since $|\boldsymbol{v}|$ is $O(|\boldsymbol{x}|^{-3})$ and $h' = p'/\rho_0 + Ts' = p'/\rho_0$ (prime denotes acoustic fluctuation). Thus we obtain an approximate representation,

$$p'(\boldsymbol{x}, t) = (\rho_0/4\pi) \int S(\boldsymbol{y}, t_{\mathrm{r}})\, \mathrm{d}^3\boldsymbol{y}. \tag{7}$$

Thus, it is found that $p'(\boldsymbol{x}, t)$ satisfies approximately the wave equation (5), when $\boldsymbol{x}$ is far from a *compact* source at $\boldsymbol{y}$. If $S(\boldsymbol{y}, t) = \mathrm{div}(\boldsymbol{\omega} \times \boldsymbol{v})$ is evaluated with the incompressible vortex motion, then the error would be $O(M^2)$.

Based on (5), Möhring [**12**] succeeded in representing the acoustic pressure p' in terms of the vorticity $\boldsymbol{\omega}$ only, and gave a mathematical basis for the term, *vortex sound*. Much earlier, Obermeier [**13**] found a formula of an acoustic wave emitted by a spinning pair of two 2D vortices.

An acoustic wave radiated by head-on collision of two vortex rings was detected experimentally by Kambe and Minota [**14**], using a pair of vortex rings generated as in Fig.2. Figure 3 shows a shadowgraph [**15**] at the time of head-on collision of two vortex rings whose velocity were much larger than the acoustic experiment [**14**]. Two vertical dark columns are the colliding vortices, and short arcs bridging them (bright-and-dark double layers) are shocklets. Visible wave patterns are weak shocks.

4.3 Dissipative structure in turbulence

In homogeneous turbulence, average rate of dissipation ε of kinetic energy $\langle \boldsymbol{v}^2 \rangle/2$ per unit mass is proportional to the average squared vorticity $\langle \boldsymbol{\omega}^2 \rangle$,

$$\varepsilon = -\frac{\mathrm{d}}{\mathrm{d}t} \tfrac{1}{2} \langle \boldsymbol{v}^2 \rangle = \nu \langle \boldsymbol{\omega}^2 \rangle, \tag{8}$$

for an incompressible fluid, where ν is the kinematic viscosity, and $\langle \cdot \rangle$ denotes ensemble average.[3] This average equation can be derived from (2) by assuming $\mathrm{div}\, \boldsymbol{v} = 0$, taking scalar product with $\boldsymbol{v}$, and replacing $\boldsymbol{F}$ with the viscous force $\boldsymbol{f}_v$. Taking average over a large volume V, integrating by parts, and omitting integrated terms over bounding surface, we obtain (8). [**16**]

In turbulence, there exists a certain straining mechanism by which vortex lines are stretched on the average. This is related to the *negative* value of the skewness S of longitudinal derivative $\partial u/\partial x$ [**16**, §8], where u is the velocity component along the x-axis, and the skewness S is defined by normalized

[3] In [**1**], Prandtl wrote that the viscous force is expressed as $\boldsymbol{f}_v = \nu \nabla^2 \boldsymbol{v} = -\nu\, \mathrm{rot}\, \boldsymbol{\omega}$ if $\mathrm{div}\, \boldsymbol{v} = 0$, hence that the vorticity $\boldsymbol{\omega}$ gets involved with the viscous diffusion. But, it is noted only that $\boldsymbol{f}_v = 0$ for $\boldsymbol{\omega} = 0$, *i.e.* irrotational flow is possible for arbitrary viscosity.

statistical average of $(\partial u/\partial x)^3$. Non-zero value of S implies that the statistics is *non-Gaussian*, and that there exists structures in turbulence, which is often called the *intermittency* [**17**]. As a thin vortex tube is stretched, its vorticity increases, and dissipation is enhanced around it in accordance with (8).

According to the theory of energy cascade of fully developed turbulence, energy is dissipated at scales of smallest eddies of order $\eta = (\nu^3/\varepsilon)^{1/4}$, called the Kolmogorov's dissipation scale. In computer simulations, the dissipative structures in turbulence are visualized as fine-scale slender objects with high level of vorticity magnitude often called *worms*.

Figure 4 shows snapshot of vorticity field [**18**] obtained by a direct numerical simulation of incompressible turbulence in a periodic box with grid points 2048^3 carried out on the *Earth Simulator*, the largest parallel computer in operation.

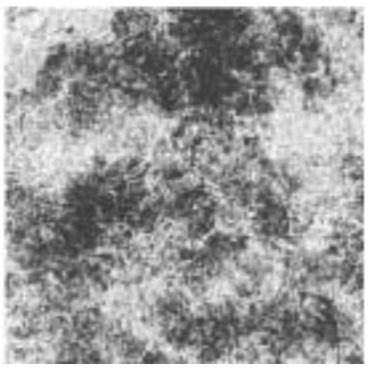
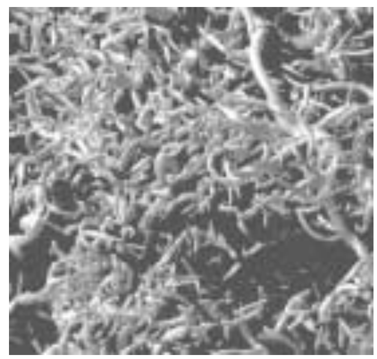
(a)  (b) 

Figure 4. High-vorticity isosurfaces obtained by DNS of 2048^3 grid points with $\nu = 4.4 \times 10^{-5}$, $\eta = 1.05 \times 10^{-3}$, the Taylor-microscale Reynolds number $R_\lambda = 732$, and the integral scale $L = 1.23$. (a) Length of a side is $2992\,\eta$, and (b) 8 times enlargement of (a), *i.e.* length of a side is $374\,\eta$, the area being 1/64 of that of (a). The isosurfaces are defined by $|\boldsymbol{\omega}| = \langle \boldsymbol{\omega} \rangle + 4\sigma$ where σ is the standard deviation of the magnitude $|\boldsymbol{\omega}|$. [**18**].

5. GAUGE FIELD

Fluid mechanics is considered as a field theory of *mass flow* in Newtonian mechanics. In the theory of gauge fields, a guiding principle is that laws of physics should be expressed in a form that is independent of any particular coordinate system. The Lagrangian of fluid flow is defined in such a way as having an invariance under Galilei transformation. Next, a gauge principle is applied to the Lagrangian, requiring it to have *symmetry*, *i.e.* the gauge invariance. In regard to the fluid flows, relevant symmetry groups are *translation group* and *rotation group* [**19**].

According to the gauge principle, time derivative of velocity is given by the covariant derivative $\nabla_t \boldsymbol{v}$ in the following form:

$$\nabla_t \boldsymbol{v} = \partial_t \boldsymbol{v} + \mathrm{grad}(\boldsymbol{v}^2/2) + \Omega \boldsymbol{v}, \qquad (\Omega \boldsymbol{v})_i = \Omega_{ij} v_j \tag{9}$$

where Ω is a linear operator called a *gauge field*, assumed to vanish in irrotational flows.

For *irrotational* flows in which $\boldsymbol{v} = \boldsymbol{v}_\mathrm{p} = \mathrm{grad}\,\phi$ (ϕ: a velocity potential), first two terms represent a time derivative which is invariant under *local translational* gauge transformation, and the expression (9) reduces to

$$\nabla_t \boldsymbol{v}_\mathrm{p} = \partial_t \boldsymbol{v}_\mathrm{p} + \mathrm{grad}(\boldsymbol{v}_\mathrm{p}^2/2) = \partial_t \boldsymbol{v}_\mathrm{p} + (\boldsymbol{v}_\mathrm{p} \cdot \nabla)\boldsymbol{v}_\mathrm{p}. \tag{10}$$

since $\frac{1}{2}\,\partial_k(\boldsymbol{v}_\mathrm{p})^2 = (\partial_i\phi)\partial_k(\partial_i\phi) = (\partial_i\phi)\partial_i(\partial_k\phi) = (\boldsymbol{v}_\mathrm{p} \cdot \nabla)(\boldsymbol{v}_\mathrm{p})_k$. There exist some liquids, in which composing particles are equivalent and indistinguishable. Local rotation of such a fluid may not be captured because local rotaion (if any) should make no difference. Therefore the flow should be inevitably *irrotational*. It is known that super-fluid flows such as He^4 (a boson) or a Bose-Einstein condensate are irrotational.

This is not the case when we consider motions of a fluid composed of distinguishable particles such as an ordinary fluid. Local rotation is distinguishable and flows are *rotational* in general. Then, it is required that the equation of fluid flows should be invariant under *rotational* gauge transformation, and that $\nabla_t \boldsymbol{v}$ is invariant with respect to Galilei transformation. From this requirement, we find [**20**] that the gauge term must be of the form $\Omega_{ij} = -\epsilon_{ijk}\omega_k$, where ω_k is the k-th component of the vorticity $\boldsymbol{\omega}$. Hence, we obtain

$$(\Omega \boldsymbol{v})_i = -\epsilon_{ijk}\omega_k v_j = (\boldsymbol{\omega} \times \boldsymbol{v})_i.$$

Thus, the vorticity $\boldsymbol{\omega}$ is found to be the gauge field, and the covariant derivative (9) is given by the material derivative, *i.e.* the *Lagrange derivative*,

$$\nabla_t \boldsymbol{v} = \partial_t \boldsymbol{v} + \mathrm{grad}(\boldsymbol{v}^2/2) + \boldsymbol{\omega} \times \boldsymbol{v} = \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \tag{11}$$

by using a well-known vector identity.

This *gauge-theoretic* formulation provides a theoretical ground to the physical analogy between the aeroacoustic interaction associated with vortices and the interaction of electron and electromagnetic-field. In the latter problem, the electromagnetic field is the gauge field.

REFERENCES

1. Ludwig Prandtl. "Über Flüssigkeitbewegung bei sehr kleiner Reibung", *Verhandlungen des III. Internationalen Mathematiker-Kongress* (Heidelberg 1904), pp.484-491, 1905.
2. Prandtl L. *Abriss der Strömungslehre*, Friedr.Vieweg & Sohn, Braunschweig, 1931.
3. Prandtl L. *Führer durch die Strömungslehre*, Friedr.Vieweg & Sohn, Braunschweig, 1965.
4. Prandtl L and Tietjens OG. *Fundamentals of Hydro- and Aeromechanics*, McGraw-Hill. (Translated from the German edition, Springer, 1931.), 1934.
5. Prandtl L and Tietjens OG. *Applied Hydro- and Aeromechanics*, McGraw-Hill. (Translated from the German edition, Springer, 1931.), 1934, [Dover].
6. Saffman PG. *Vortex Dynamics*, Cambridge Univ. Press, 1992.
7. Minota T. "Interaction of a shock wave with a high-speed vortex ring", *Fluid Dyn. Res.* **12**, pp.335-342, 1993.
8. von Karman T. *Göttingen Nachrichten*, Göttingen, pp.547-556, 1912.
9. Bjerkness V. "Lectures on hydrodynamic forces acting at a distance" (German), Leipzig, 1900-1902.
10. Powell A. "Theory of vortex sound", *J. Acoust. Soc. Am.* **36**, 177-195, 1964.
11. Howe MS. "Contributions to the theory of aerodynamic sound", *J. Fluid Mech.* **71**, pp.625-973, 1975.
12. Möhring W. "On vortex sound at low Mach number", *J. Fluid Mech.* **85**, pp.685-691, 1978.
13. Obermeier F. "Berechnung aerodynamisch erzeugter Shallfelder mittels der Methode der Matched Asymptotic Expansions", *Acustica* **18**, 238-240, 1967.
14. Kambe T and Minota T. "Acoustic wave radiated by head-on collision of two vortex rings", *Proc. R. Soc. Lond. A* **386**, pp.277-308, 1983.
15. Minota T, Nishida M and Lee MG. "Head-on collision of two compressible vortex rings", *Fluid Dyn. Res.* **22**, pp.43-60, 1998.
16. Batchelor GK. *The Theory of homogeneous turbulence*, Cambridge Univ. Press, 1953.
17. Kambe T. "Statistical laws governed by vortex structures in fully developed turbulence", in *IUTAM Symposium on Geometry and Statistics of Turbulence* (eds: Kambe, Nakano & Miyauchi, held in 1999), Kluwer Academic Pub., Dordrecht, pp.117-126, 2001.
18. Yokokawa M, Itakura K, Uno A, Ishihara T, Kaneda Y."16.4TFlops direct numerical simulation of turbulence by a Fourier spectral method on the Earth Simulator", *Proc. IEEE/ACM SC2002 Conf.*, Baltimore, 2002 (http://www.sc-2002.org/paperpdfs/pap.pap273.pdf);
    Kaneda Y, Ishihara T, Yokokawa M, Itakura K, Uno A. "Energy dissipation rate and energy spectrum in high resolution direct numerical simulations of turbulence in a periodic box", *Physics of Fluids*, **15** (2), pp.L21-L24, 2003.
19. Kambe T. *Geometrical Theory of Dynamical Systems and Fluid Flows*, Ch.7, World Scientific Publishing Co., 2004;
    Kambe T."Gauge principle and variational formulation of ideal fluid flows", *Preprint* TK2004a (http://hw001.gate01.com/~kambe).
20. Kambe T. "Gauge principle and variational formulation for flows of an ideal fluid", *Acta Mechanica Sinica* **19**, pp.437-452, 2003;
    Kambe T. "Gauge principle for flows of an ideal fluid", *Fluid Dyn. Res.* **32**, pp.193-199, 2003.